A Search for a Magic Hourglass

Duncan A. Buell
Department of Computer Science and Engineering
University of South Carolina
Columbia, South Carolina 20209
buell@cse.sc.edu

July 1, 2004

Abstract

An open problem mostly of a recreational nature is to find a 3×3 magic square all of whose entries are themselves square integers, or, failing in that search, to find a magic "hourglass" all of whose entries are square integers. We describe an unsuccessful search for such an hourglass.

1 Introduction

An old problem largely of a recreational nature is to find a 3×3 magic square all of whose entries are squares of integers. A subproblem is to find a magic "hourglass"

$$\begin{array}{ccc} a-b & a+b+c & a-c \\ & a & \\ a+c & a-b-c & a+b \end{array} \tag{1}$$

all of whose entries are squares of integers.

We describe below an unsuccessful search for such an hourglass, and we announce the following computational result.

Theorem 1.1. *There is no magic hourglass for which a is less than $25 \cdot 10^{24}$.*

2 The Parameterization, and a Computation

The first relatively easy parameterization is to note that we must have

$$a = (m^2 + n^2)^2 = (r^2 + s^2)^2 = (u^2 + v^2)^2 \tag{2}$$

in integer variables m, n, r, s, u, v, and that with this parameterization and the assignment

$$\begin{aligned} b &= 4mn(m^2 - n^2) \\ c &= 4rs(r^2 - s^2) \\ d &= 4uv(u^2 - v^2) \end{aligned} \tag{3}$$

we will have satisfied all the requirements for

$$\begin{matrix} a-b & a-d & a-c \\ & a & \\ a+c & a+d & a+b \end{matrix}$$

to be a magic hourglass as desired except for the requirement that the two horizontals sum to $3a$, that is, that

$$b + c = -d. \tag{4}$$

Our search for a magic hourglass, therefore, begins by finding all values of $A = \sqrt{a}$ that can be written in three or more distinct ways as a sum of squares. We then try a minimum set of permutations of $S = \{m, n, r, s, u, v\}$, with signs, in an attempt to satisfy (4).

The code for the search was written in C and run under MPI, in the background at very low priority, on multiple processors of the SGI Challenge computer at CCS through most of calendar year 1998. Up through $A = 5 \cdot 10^{12}$, however, no solution to (1) was found.

In a computation such as this, when one is searching the proverbial haystack for a needle, it is easy for a null result to be obtained incorrectly through an error in programming. To guard against this, the program must be written not only to be self-checking, but also so as to produce *some* sort of output that can be examined and tested for correctness. Our computation, therefore, proceeded in three stages.

First we enumerated the sums of squares in a given block of integers. Those integers A with at least three such representations were kept active. Next, we tested the solutions S to (2) testing (4) not as an equation but as a congruence modulo 2^{16}. This pared down the list to be tested to a manageable size and could be done using only single-precision (64-bit) arithmetic, but by only testing the low order bits we avoided getting an erroneous null result. The survivors of this second filter were then tested using `gmp` multiprecise arithmetic, again not by testing (4) for equality but by testing for the highest power of 2 satisfied by the solution S. Various tests along the way kept counts of the number of survivors of the three filtering steps so that the computation could proceed almost without supervision but nonetheless be robust enough to generate error messages in the event of power failures, disk partitions filled to capacity, and the like.

For example, we find that

$$\begin{aligned}
1193162282546 &= 1043815^2 + 321889^2 = 1066211^2 + 237395^2 = 1076911^2 + 182825^2 \\
2727247005314 &= 1456705^2 + 777983^2 = 1651267^2 + 23755^2 = 1643875^2 + 157867^2 \\
2726033914369 &= 1245313^2 + 1084080^2 = 1643860^2 + 154137^2 = 1613863^2 + 348540^2
\end{aligned}$$

and for the first example we may take

$$\{m, n, r, s, u, v\} = \{1043815, -321889, 1066211, 237395, 1076911, 182825\}$$

to obtain

$$\begin{aligned}
b &= -1325070315895532182090560 \\
c &= 1093903960779403233924480 \\
d &= 887021653181324514532800
\end{aligned}$$

.

We then find that

$$b + c + d = 655855298065195566366720$$

which is zero modulo 2^{46} but not modulo 2^{47}. A similar congruence holds modulo 2^{46} but not 2^{47} for the other two cases above, using

$$\{m, n, r, s, u, v\} = \{1456705, 777983, 1651267, 23755, 1643875, -157867\}$$

and

$$\{m, n, r, s, u, v\} = \{1245313, -1084080, 1643860, 154137, 1613863, 348540\}.$$

These were the three examples found for which (4), viewed as a congruence, was solvable for the highest power of 2.

For completeness, we also give a table of the number of solutions to (4) as a congruence, for all powers of 2 greater than or equal to 30.

Power	Number of Solutions
30	107491
31	53616
32	26918
33	13315
34	6768
35	3342
36	1699
37	825
38	428
39	219
40	94
41	62
42	22
43	18
44	9
45	5
46	3