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HOW MANY SQUARES ARE THERE, Mr TARRY?

ABSTRACT

In this paper, I enumerate all the 8x8 bimagic squares given by the Tarry's pattern. This pattern is a 8x8 Greco-Latin magic pattern and I show that in fact we can find 14,784 different 8x8 Greco-Latin magic patterns of a similar type. Among these 14,784 patterns, only some ones give bimagic squares and I calculate the total number of 8x8 bimagic squares generated with this method.

THE TARRY'S PATTERN

In his article ref. [1], Tarry gives the pattern of a 8x8 pandiagonal magic square which has all the rows and all the columns bimagic:

a p+r	b-c q-r+s	b+d p	a+c+d q+s	b p+r+s	a+c q-r	a+d p+s	b-c+d q
b p	a+c q+s	a+d p+r	b-c+d q-r+s	a p+s	b-c q	b+d p+r+s	a+c+d q-r
a+c+d p+r+s	b+d q-r	b-c p+s	a q	b-c+d p+r	a+d q-r+s	a+c p	b q+s
b-c+d p+s	a+d q	a+c p+r+s	b q-r	a+c+d p	b+d q+s	b-c p+r	a q-r+s
a+d q-r	b-c+d p+r+s	b q	a+c p+s	b+d q-r+s	a+c+d p+r	a q+s	b-c p
b+d q	a+c+d p+s	a q-r	b-c p+r+s	a+d q+s	b-c+d p	b q-r+s	a+c p+r
a+c q-r+s	b p+r	b-c+d q+s	a+d p	b-c q-r	a p+r+s	a+c+d q	b+d p+s
b-c q+s	a p	a+c+d q-r+s	b+d p+r	a+c q	b p+s	b-c+d q-r	a+d p+r+s

The 8 numbers a, a+c, b-c, b, a+d, a+c+d, b-c+d, b+d have to be chosen among (1,2,3,…,8) and the 8 numbers p, p+r, q-r, q, p+s, p+r+s, q-r+s, q+s among (0,8, 16,…,56).
The two main diagonals are also bimagic if the supplementary condition

$$r(a-b)=c(p-q)$$

is satisfied. For example if: a=1, b=4, c=1, d=4, p=0, q=24, r=8, s=32.

Tarry writes a little rapidly there is an "infinity" of squares which can be constructed with this pattern. In fact, it is wrong. The exact number can be easily calculated by tabulation, i.e. when giving to the parameters all the possible values satisfying the conditions indicated by Tarry.

I found there are

2,304 semi-bimagic squares

320 bimagic squares

(I call "semi-bimagic" the magic squares which have all the rows and all the columns bimagic).

I studied particularly the 320 bimagic solutions. There are only 80 unique squares because the group (I, V, H, R2) is working on this set (see my site ref [4] for the notations).

All these bimagic squares are Greco-Latin, pandiagonal of type complete and with the 2 semi-diagonals bimagic (the semi-diagonals are A4, B3, C2,…, H5 and A5, B6, C7,…, H4).

Each elementary square is Latin diagonal, self-orthogonal (orthogonal to its transposed) and pandiagonal (but not Latin pandiagonal ; the two semi-diagonals are nevertheless Latin).

I have printed all these squares. For example, here is the first bimagic one, which is made with the above-mentioned parameters of Tarry:

9	51	8	62	44	18	37	31
4	58	13	55	33	27	48	22
46	24	35	25	15	53	2	60
39	29	42	20	6	64	11	49
21	47	28	34	56	14	57	3
32	38	17	43	61	7	52	10
50	12	63	5	19	41	30	40
59	1	54	16	26	36	23	45

The classical basis with the numbers (1, 2, 3,…, 8) and (0, 8, 16,…, 56) was here used by Tarry. But, other basis are possible: 20 basis exactly (see my site ref [4]).

I remember the 10 first basis:

1, 2, 3, 4, 5, 6, 7, 8 ; 0, 8,16,24,32,40,48,56
1, 2, 3, 4, 9,10,11,12 ; 0, 4,16,20,32,36,48,52
1, 2, 3, 4,17,18,19,20 ; 0, 4, 8,12,32,36,40,44
1, 2, 3, 4,33,34,35,36 ; 0, 4, 8,12,16,20,24,28
1, 2, 5, 6, 9,10,13,14 ; 0, 2,16,18,32,34,48,50
1, 2, 5, 6,17,18,21,22 ; 0, 2, 8,10,32,34,40,42
1, 2, 5, 6,33,34,37,38 ; 0, 2, 8,10,16,18,24,26
1, 2, 9,10,17,18,25,26 ; 0, 2, 4, 6,32,34,36,38
1, 2, 9,10,33,34,41,42 ; 0, 2, 4, 6,16,18,20,22
1, 2,17,18,33,34,49,50 ; 0, 2, 4, 6, 8,10,12,14

I found that the Tarry's pattern with all the basis gives:

46,080 semi-bimagic squares, i.e. 2,304 for each basis from 1 to 20,

3,584 bimagic squares, distributed as follows:

Basis	Nb of sq.
1	320
2	192
3	128
4	128
5	128
6	320
7	128
8	192
9	128
10	128
11	128
12	128
13	192
14	128
15	320
16	128
17	128
18	128
19	192
20	320
Total	3,584

Among the 3,584 bimagic squares, there are 3,584/4 = 896 unique squares. All are Greco-Latin (with the indicated basis) and pandiagonal of type complete.

I have printed all these squares. For example, here is the first bimagic square with the second basis:

```
 17  39  12  62  24  34  13  59
  4  54  25  47   5  51  32  42
 30  44   7  49  27  45   2  56
 15  57  22  36  10  64  19  37
 41  31  52   6  48  26  53   3
 60  14  33  23  61  11  40  18
 38  20  63   9  35  21  58  16
 55   1  46  28  50   8  43  29
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The two semi-diagonals of this square are magic but not bimagic.

There is an isomorphism between the 320 squares of the first basis and the 320 squares of the 20^{th} basis, and more generally between the squares of one basis and the squares of the "corresponding" basis.

> The explanation lies in the fact that we pass from one basis to the corresponding basis by exchange between the low and the high component square (see my site) and that there is here – between these two elementary squares - the geometric transformation $(16745238)_{all}$*G, with a convenient permutation after (see next paragraph). The geometric transformation $(16745238)_{all}$*G applies then the squares of one basis on the squares of the corresponding basis.

GENERALIZATION

I inspected the 320 bimagic squares coming from the Tarry's pattern with the first basis. All these squares have the same structure:
- the low component square is a variation of the sq. # 454

1	2	3	4	5	6	7	8
5	6	7	8	1	2	3	4
4	3	2	1	8	7	6	5
8	7	6	5	4	3	2	1
7	8	5	6	3	4	1	2
3	4	1	2	7	8	5	6
6	5	8	7	2	1	4	3
2	1	4	3	6	5	8	7

- the high component square is a variation of the sq # 49

1	2	3	4	5	6	7	8
3	4	1	2	7	8	5	6
5	6	7	8	1	2	3	4
7	8	5	6	3	4	1	2
6	5	8	7	2	1	4	3
8	7	6	5	4	3	2	1
2	1	4	3	6	5	8	7
4	3	2	1	8	7	6	5

The squares # 454 and # 49 belong to the list of the 1,152 diagonal Latin squares of order 8 which are self-orthogonal and which have 1,2,3,4,5,6,7,8 on the first row. Cf my enumeration of some 8x8 Greco-Latin magic squares, ref [3].

A Greco-Latin square can then be made with the sq. # 454 and # 49 which are orthogonal. These two squares belong to the family of the sq #2 (I remember I showed that all the 1,152 squares can be generated from 8 basic elementary squares #1, #2, #3, #4, #6, #8, #10 and #13 by applying the 1,536 geometric transformations of G 1,536 and the 40,320 permutations on the 8 first numbers).

The family of the sq #2 is made of 96 squares (out of 1152).

We can then try <u>to generalize the Tarry's pattern by searching other couples of diagonal Latin squares for building a Greco-Latin square. Each Greco-Latin pattern generates 20*(8!)*(8!) squares. We can search after, among all these squares, those which are bimagic.</u>

I made this task on the self-orthogonal diagonal Latin squares.

Among the 1,152 elementary diagonal Latin squares which are self-orthogonal, I found there are 14,784 couples of squares which are orthogonal or Greco-Latin:

there are 14,784 different diagonal Greco-Latin patterns made from 8x8 self-orthogonal diagonal Latin squares

(each elementary square has several orthogonal squares in the list ;
among these 14,784 couples, we have naturally the 1,152 couples made with one square and its transposed ; these 1,152 couples never give bimagic solutions).
But bimagic solutions are only found with couples made with 2 squares coming from the 96 squares of the family of the sq #2 (there are 1,152 orthogonal such couples, and finally only 864 orthogonal couples give bimagic solutions. More exactly, each square out of 96 has 12 orthogonal squares, but each square has only 9 bimagic orthogonal squares, and 9*96=864).

> there are 864 different bimagic diagonal Greco-Latin patterns made from 8x8 self-orthogonal diagonal Latin squares

In my enumeration of couples, I count for 2 solutions the couples x-y and y-x.
I have a file with the 14,784 couples and the 14,784 diagonal Greco-Latin patterns. I have defined a “standard position” of a Greco-Latin square by a magic square with the first basis and with 1,10,19,…, 64 on the first row.

I found 552,960 bimagic squares with the first basis. The group G 1,536 is working on this set and then we can reduce the number to 552,960/1,536 = 360 elementary (or essentially different) squares. I have printed these 552,960 and 360 solutions.

For all the basis, I found 2,016 elementary bimagic solutions by a reduced program/1536 with 10 runs:

Basis	Number of elementary sq.
1	360
2	216
3	144
4	144
5	144
6	360
7	144
8	216
9	144
10	144
Total	2,016

We have a similar distribution (according to the basis) as for the above-mentioned Tarry’s squares tabulation: the same number for basis #1 and 6, for basis #2 and #8, and for basis #3, 4, 5, 7, 9, 10. And naturally the same number for a basis and for its corresponding basis.
But the squares of one basis are here identical to the squares of the corresponding basis, because there is a transformation of G 1,536 between the 2 squares of each couple of Latin squares, and when we exchange these 2 squares, we find the same resulting elementary square.

I have printed all these 2,016 solutions in 10 files.

Unfortunately, all these 2,016 squares are not different: they are different in each file, but a same square can appear in 3 different files out of 10.

I have filtered all these 2,016 squares, and I found 1,344 different squares: 1,008 squares appear only 1 time in 1 file and 336 squares appear 3 times in 3 different files (2,016=1,008+3*336=1,344+2*336=3/2*1344). These 1344 squares are printed in a file with the identification of each square among the 10 source files

The total number of bimagic squares generated by this method is then

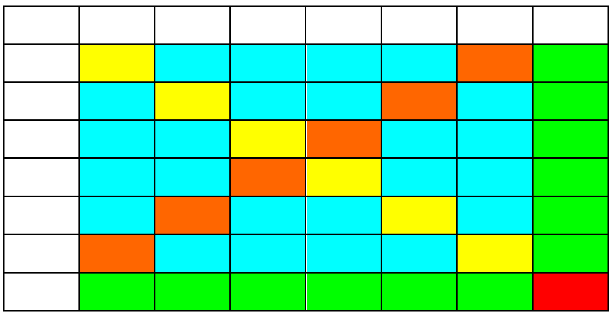
1,344*1,536 = 2,064,384 when counting all the squares

or 1,344*192 = 258,048 unique bimagic squares

TYPES OF THE BIMAGIC SOLUTIONS

I inspected the elementary bimagic and the unique bimagic solutions.

There are 97 types of squares, distributed in 5 families defined in an abbreviated form by A1+X=65, X being one of the 49 coloured cell in the figure hereafter (in fact the pattern is also necessary for defining a type and there are 2 types for a given X, except for X=H8):



Here are the 5 families:

Red (1 cell)……………….. …………..1 type (associative)
Yellow (6 cells)……………………... 12 types (complete and isomorphic types)
Orange (6 cells)……….. …………….12 types
Green (12 cells)………………………24 types
Blue (24 cells)………………………..48 types
Total (49 cells) 97 types

I count 2 types for 2 cells symmetrical by the first diagonal (instead of only one Dudeney's type).

Here is the distribution of the squares into the 5 families:

Family	Total	Nb of uniques (x192)	Nb of types	Nb of uniques by type
Red	112	21,504	1	21,504
Yellow	224	43,008	12	3,584
Orange	336	64,512	12	5,376
Green	224	43,008	24	1,792
Blue	448	86,016	48	1,792
Total	1,344	258,048	97	

We find 21,504 associative unique squares out of the total of 161,472 found by Walter Trump
and 3,584 complete unique squares “ “ 29,376 “ “

CONCLUSION

I enumerated all the squares coming from the Tarry’s pattern and I showed a method for finding a particular kind of bimagic squares: the squares coming from Greco-Latin squares. This method was applied to the 8x8 diagonal Latin squares which are self-orthogonal.

An extension to all the 8x8 diagonal Latin squares can be tried, but the task is surely very hard. Idem for an extension to orders superior to 8 (I remember that the order 8 is the minimum order for finding bimagic squares).

When carrying through this study, I found an accessory result: there are 14,784 different diagonal Greco-Latin patterns made from 8x8 self-orthogonal diagonal Latin squares. And each pattern generates 20*(8!)*(8!) squares.

ACKNOWLEDGEMENTS

Thank you to Aale de Winkel who pointed out that my 2,016 elementary bimagic squares were not all different and who found the number of 1,344 for the reduced set. He gave me also the principle of a filtering program. I found the same number by a program which gives besides the identification of each square in each basis.

REFERENCES

[1] Gaston Tarry, Compte-rendu de l’AFAS, 1903, p.141

[2] Jacques Bouteloup, Carrés magiques, carrés latins et eulériens, Choix 1991
See pages 152-154 for Tarry and pages 114-116 for a Greco-Latin square.
On page 152, there is a formula for the bimagic condition on a Greco-Latin square.

[3] Francis Gaspalou, enumeration of some 8x8 Greco-Latin magic squares (mail of July 18, 2010 and mail of October 5, 2010)

[4] Site http://www.gaspalou.fr/magic-squares/