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THERE ARE NO COMPACT BIMAGIC SQUARES

We remember the definition of a compact square: the sum of the 4 numbers in every 2x2 subsquare is constant.

1. METHOD OF DEMONSTRATION FOR THE ORDER 8 (by Francis Gaspalou)

In May 2011, F. Gaspalou enumerated all the 8x8 compact semi-magic squares. It is then easy to check with a program that none of these squares is bimagic. And if there is none 8x8 compact semi-magic square, a fortiori there is none 8x8 compact magic square.

But a direct proof can be given.

Let consider the 8 numbers A1, A2,…,A8 on the first row.

For the numbers of the second row, it is easy to see that :
A1+A2+B1+B2=130 => B2=130-A1-A2-B1
A2+A3+B2+B3=130 => B3=130-A2-A3-B2=A1-A3+B1
A3+A4+B3+B4=130 => B4=130-A3-A4-B3=130-A1-A4-B1
A4+A5+B4+B5=130 => B5=130-A4-A5-B4=A1-A5+B1
etc
i.e. all the numbers of the second row can be calculated with the numbers of the first row and with B1.

If we write that $B1^2 + B2^2 + B3^2 + \ldots + B8^2 = 11180$
We have $B1^2 + (130\text{-}A1\text{-}A2\text{-}B1)^2 + (A1\text{-}A3\text{+}B1)^2 + \ldots + (130\text{-}A1\text{-}A8\text{-}B1)^2 = 11180$
If A1, A2,…,A8 are given, then B1 is solution of an equation of degree 2.

In a same way, the relation $D1^2 + D2^2 + D3^2 + \ldots + D8^2 = 11180$
can be written $D1^2 + (130\text{-}A1\text{-}A2\text{-}D1)^2 + (A1\text{-}A3\text{+}D1)^2 + \ldots + (130\text{-}A1\text{-}A8\text{-}D1)^2 = 11180$
which is exactly the same equation of degree 2 where we replace B1 by D1.

Idem for F1 and for H1.

Then it is possible to find B1 and D1 in function of A1, A2,...,A8 but not F1 nor H1 because an equation of degree 2 can have only 2 solutions.

It is then impossible to build a compact semi-magic square of order 8 which is bimagic.

2. GENERALIZATION FOR ANY ORDER (by Walter Trump)

The previous demonstration was generalized to

- *the compact semi-magic squares of all orders* [W. Trump remarked first that this demonstration could be generalized to the compact semi-magic squares of even order superior to 4. And he showed after that for n=4 it is also impossible to find a solution with distinct numbers (see his demonstration in annex). At last, it is well known that there are no compact semi-magic squares of odd orders]

- *not consecutive but distinct numbers.*

CONCLUSION: Compact bimagic squares with distinct numbers do not exist.

ANNEX

A COMPACT SEMI-MAGIC SQUARE OF ORDER 4 WITH DISTINCT INTEGERS IS NOT BIMAGIC (by Walter Trump)

Proof:

$S :=$ magic constant = sum of each 2x2-square

$S = A1 + A2 + A3 + A4$

For compact squares the following equations are known (easy to prove):

$A1 + B1 + A2 + B2 = S \;\Rightarrow\; B2 = S - A1 - A2 - B1 = A3 - B1 + A4$

$A1 + B1 = A3 + B3 \;\Rightarrow\; -B3 = A3 - B1 - A1$

$A1 + B1 + A4 + B4 = S \;\Rightarrow\; B4 = S - A1 - A4 - B1 = A3 - B1 + A2$

Assumption: the square is bimagic with bimagic constant S_2.

$A1^2 + A2^2 + A3^2 + A4^2 = S_2$

$B1^2 + B2^2 + B3^2 + B4^2 = S_2$

$B1^2 + B2^2 + (-B3)^2 + B4^2 = S_2$

$B1^2 + [(A3 - B1) + A4]^2 + [(A3 - B1) - A1]^2 + [(A3 - B1) + A2]^2 = S_2$

$B1^2 - A3^2 + A3^2 + [(A3 - B1) + A4]^2 + [(A3 - B1) - A1]^2 + [(A3 - B1) + A2]^2 = S_2$

$-(A3 - B1)\cdot(A3 + B1) + A3^2 + A4^2 + A1^2 + A2^2 + 3\cdot(A3 - B1)^2 + 2\cdot(A3 - B1)\cdot(A4 - A1 + A2) = S_2$

$-(A3 - B1)\cdot(A3 + B1) + S_2 + 3\cdot(A3 - B1)^2 + 2\cdot(A3 - B1)\cdot(A4 + A2 - A1) = S_2$

$-(A3 - B1)\cdot(A3 + B1) + 3\cdot(A3 - B1)^2 + 2\cdot(A3 - B1)\cdot(S - A3 - 2\cdot A1) = 0$

As A3 and B1 are distinct integers their difference $(A3 - B1)$ is not 0. $\Rightarrow$

$-(A3 + B1) + 3\cdot(A3 - B1) + 2\cdot(S - A3 - 2\cdot A1) = 0$

$-4\cdot B1 + 2\cdot S - 4\cdot A1 = 0$

$B1 + A1 = \frac{1}{2} S$

(B1 and A1 are complementary numbers.)

Symmetry consideration:

Interchange x- and y-coordinates and do the same investigations for A2 instead of B1.

$\Rightarrow\; A2 + A1 = \frac{1}{2} S$

$\Rightarrow\; A2 = B1$ This stands in contradiction to distinct integers.

$\Rightarrow$ The square cannot be bimagic. q.e.d.

Further considerations show that a semi-magic 4x4-square that is compact and bimagic can only consist of two different integers a, b. And this square is not magic as long as $a \neq b$.

a	b	a	b
b	a	b	a
a	b	a	b
b	a	b	a