# Estimating the number of $K$-multimagic $\alpha$-hypercube series

In this note I want to show that the leading-order Gaussian estimate for the number of $K$-multimagic series for $\propto$-dimensional (hyper)cubes ($\alpha \geq 3$) of order $N$ can be expressed as follows:

$$\mathcal{N}_{\propto,g}^{(K)}(N) \approx \sqrt{\frac{(2K+1)!}{(2\pi)^{K+1}} \prod_{i=1}^{K} \frac{(K+i)!}{i!}} \cdot \frac{(N^{\alpha-1}e)^N}{N^{(\alpha K+1)(K+1)/2}} \ (N \to \infty)$$

The formula for squares ($\alpha = 2$) is slightly different because it has an extra constant factor $1/\sqrt{e}$. Some special cases ($K \leq 3$ and $\alpha \leq 3$) already appear in M. Quist's paper (arXiv:1306.0616). This note applies the ideas already presented in his paper in order to derive this more general formula. Although there are some minor additional problems to be solved, no really new ideas are necessary.

In general, consider $m$ (K+1)-vectors $S_i = (s_{i,0}, \dots, s_{i,K})$ where $s_{i,k} = \binom{i}{k}$, and then consider $m$ pairwise independent random vectors $X_i = (x_{i,0}, \dots, x_{i,K})$ equal to $S_i$ with probability $\beta$ and equal to $(0, \dots, 0)$ otherwise.

Then the distribution of the vector $A = \sum_{i=1}^{m} X_i$ approaches a (K+1)-dimensional Gaussian as $m$ becomes larger. Taking $m = N^\alpha$ and $\beta = 1/N^{\alpha-1}$ we have, as M. Quist shows,

$$\mathcal{N}_{\propto,g}^{(K)}(N) = \beta^{-\beta m}(1-\beta)^{-(1-\beta)m} \cdot P_g^*(m,\beta)$$

where $P_g^*(m,\beta)$ is the "central probability" (the peak value) of the above (K+1)-dimensional Gaussian distribution. From its well-known probability density function, see for example https://en.wikipedia.org/wiki/Multivariate_normal_distribution, it is easy to see that

$$P_g^*(m,\beta) \approx \frac{1}{\sqrt{(2\pi)^{K+1} \det \Sigma}}$$

where $\Sigma$ is the covariance matrix. If we write $A = (a_0, \dots, a_K)$ with $a_k = \sum_{i=1}^{m} x_{i,k}$, then

$$\Sigma = \begin{pmatrix} cov(a_0, a_0) & \cdots & cov(a_0, a_K) \\ \vdots & \ddots & \vdots \\ cov(a_K, a_0) & \cdots & cov(a_K, a_K) \end{pmatrix}$$

Because the $X_i$ are pairwise independent,

$$cov(a_j, a_k) = \sum_{i=1}^{m} cov(x_{i,k}, x_{i,k}) = \sum_{i=1}^{m} \left(E[x_{i,j}x_{i,k}] - E[x_{i,j}]E[x_{i,k}]\right)$$

and this can be worked out as follows:

$$cov(a_j, a_k) = \sum_{i=1}^{m} (\beta s_{i,j}s_{i,k} - (\beta s_{i,j})(\beta s_{i,k})) = \beta(1-\beta)\sum_{i=1}^{m} s_{i,j}s_{i,k} = \beta(1-\beta)\sum_{i=1}^{m} \binom{i}{j}\binom{i}{k}$$

Use the two approximations

$$\binom{i}{t} \approx \frac{i^t}{t!}\ (i \to \infty) \text{ and } \sum_{i=1}^{m} i^t \approx \frac{m^{t+1}}{t+1}\ (m \to \infty)$$

to obtain

$$cov(a_j, a_k) \approx \frac{\beta(1-\beta)}{j!\,k!} \sum_{i=1}^{m} i^{j+k} \approx \frac{\beta(1-\beta)}{j!\,k!} \frac{m^{j+k+1}}{j+k+1} = \beta(1-\beta)m \frac{m^{j+k}}{j!\,k!\,(j+k+1)}$$

To compute $\det \Sigma$, note that all elements on the same row as the element $cov(a_j, a_k)$ contain a factor $m^j/j!$, and that at the same time all elements on the same column as the element $cov(a_j, a_k)$ contain another factor $m^k/k!$. So

$$\det \Sigma \approx (\beta(1-\beta)m)^{K+1} \left( \prod_{i=1}^{K} \frac{m^i}{i!} \right)^2 \det H_{K+1}$$

where $H_{K+1}$ is the Hilbert matrix of order $K+1$. Its determinant is well-known (an inductive proof can be found in [http://s3.amazonaws.com/cramster-resource/10732_n_23997.pdf](http://s3.amazonaws.com/cramster-resource/10732_n_23997.pdf)):

$$\det H_n = \frac{(n-1)!!^4}{(2n-1)!!}$$

where $n!! := \prod_{i=1}^{n} i!$. So

$$\det \Sigma \approx (\beta(1-\beta)m)^{K+1} \left( \prod_{i=1}^{K} \frac{m^i}{i!} \right)^2 \frac{K!!^4}{(2K+1)!!} = (\beta(1-\beta)m)^{K+1} \left( \prod_{i=1}^{K} m^i \right)^2 \frac{K!!^2}{(2K+1)!!}$$

Working out the factors separately,

$$\beta m(1-\beta) = N(1 - 1/N^{\alpha-1}) \approx N\ (N \to \infty)$$

$$\prod_{i=1}^{K} m^i = m^{1+2+\cdots+K} = m^{K(K+1)/2} = N^{\alpha K(K+1)/2}$$

$$\frac{K!!^2}{(2K+1)!!} = \frac{\left(\prod_{i=1}^{K} i!\right)^2}{\left(\prod_{i=1}^{K} i!\right)\left(\prod_{i=K+1}^{2K} i!\right)(2K+1)!} = \frac{\prod_{i=1}^{K} i!}{\left(\prod_{i=K+1}^{2K} i!\right)(2K+1)!} = \frac{1}{(2K+1)!} \prod_{i=1}^{K} \frac{i!}{(K+i)!}$$

one obtains the following approximation for $\det \Sigma$ and $P_g^*(m,\beta)$:

$$\det \Sigma \approx N^{K+1} N^{\alpha K(K+1)} \frac{1}{(2K+1)!} \prod_{i=1}^{K} \frac{i!}{(K+i)!} = N^{(\alpha K+1)(K+1)} \frac{1}{(2K+1)!} \prod_{i=1}^{K} \frac{i!}{(K+i)!}$$

$$P_g^*(m,\beta) \approx \frac{1}{\sqrt{(2\pi)^{K+1} \det \Sigma}} \approx \frac{1}{N^{(\alpha K+1)(K+1)/2}} \sqrt{\frac{(2K+1)!}{(2\pi)^{K+1}} \prod_{i=1}^{K} \frac{(K+i)!}{i!}}$$

To obtain the approximation for $\mathcal{N}_{\propto,g}^{(K)}(N)$ appearing in the beginning of this note we only have to insert an approximation for the factor $\beta^{-\beta m}(1-\beta)^{-(1-\beta)m}$, where $m = N^{\alpha}$ and $\beta = 1/N^{\alpha-1}$. M. Quist already showed that for $\propto = 2$

$$\beta^{-\beta m}(1-\beta)^{-(1-\beta)m} \approx (Ne)^N/\sqrt{e} \ (N \to \infty)$$

and that for $\propto \geq 3$

$$\beta^{-\beta m}(1-\beta)^{-(1-\beta)m} \approx (N^{\propto-1}e)^N \ (N \to \infty)$$

Anyway, here is a perhaps slightly more detailed derivation.

$$\begin{aligned}\beta^{-\beta m}(1-\beta)^{-(1-\beta)m} &= N^{(\alpha-1)N}(1-1/N^{\alpha-1})^{-(N^{\alpha}-N)} \\ &= N^{(\alpha-1)N}\exp[-(N^{\alpha}-N)\ln(1-1/N^{\alpha-1})] \\ &= N^{(\alpha-1)N}\exp[((N^{\alpha}-N)/N^{\alpha-1})(-\ln(1-1/N^{\alpha-1})/(1/N^{\alpha-1}))]\end{aligned}$$

Now consider the following Taylor series expansion:

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots \ (-1 < y < 1)$$

Integrate both sides, and then divide both sides by $y$:

$$-\frac{1}{y}\ln(1-y) = 1 + \frac{1}{2}y + \frac{1}{3}y^2 + \cdots \ (0 < y < 1)$$

Apply this expansion with $y = 1/N^{\alpha-1}$:

$$\beta^{-\beta m}(1-\beta)^{-(1-\beta)m} = N^{(\alpha-1)N}\exp\left[N\left(1-\frac{1}{N^{\alpha-1}}\right)\left(1+\frac{1}{2N^{\alpha-1}}+\cdots\right)\right]$$

For $\propto = 2$ the argument of the exponential is

$$N\left(1-\frac{1}{N}\right)\left(1+\frac{1}{2N}+\cdots\right) = (N-1)\left(1+\frac{1}{2N}+\cdots\right) = \left(N+\frac{1}{2}+\cdots\right)-\left(1+\frac{1}{2N}+\cdots\right) \approx N-\frac{1}{2}$$

So after substituting:

$$\beta^{-\beta m}(1-\beta)^{-(1-\beta)m} \approx N^N e^{N-1/2} = (Ne)^N/\sqrt{e}$$

For $\propto \geq 3$ there is no constant term in the exponent, so the result becomes

$$\beta^{-\beta m}(1-\beta)^{-(1-\beta)m} \approx N^{(\alpha-1)N}e^N = (N^{\alpha-1}e)^N$$

This completes the derivation of the general formula in the beginning of this note. There is one more thing I would like to mention at this point. When I computed the exact matrix $\Sigma$ and its determinant $\det\Sigma$, I noticed a pattern. For example with $K = 3$ an exact symbolic computation shows that

$$\det\Sigma = \frac{1}{870912000}(\beta(1-\beta)m)^4\,(m^2-1)^3\,(m^2-2)^2\,(m^2-3)$$

This suggests a surprisingly simple general formula (simple compared to the complexity of $\Sigma$ itself):

$$\det \Sigma = \frac{K!!^2}{(2K+1)!!}\,(\beta(1-\beta)m)^{K+1}\prod_{i=1}^{K}(m^2-i)^{K+1-i}$$

However, I did not try to find a general proof. Such a proof does not seem to be trivial (at least not to me), and it is not needed in this note because only the leading term of this polynomial in $m$ is required.

Finally some remarks about the agreement with the known exact results.

M. Quist already mentions in his paper that the results for trimagic series will not be very accurate for several reasons. I quote: "The agreement is fairly poor at the accessible values of $N$, and because the errors oscillate in magnitude, the $1/N$ correction cannot improve matters much. There are clearly non-perturbative effects that need to be better understood, and these effects are evidently more important for higher multimagic series. Indeed, similar (albeit smaller) oscillatory effects are already apparent in the bimagic series approximation."

I'll just illustrate this with another example: the number of trimagic series for *cubes*, for which only a few exact results are knows today (see http://www.multimagie.com/English/CubeSeries.htm).

$$\mathcal{N}_{3,g}^{(3)}(N) \approx \frac{720\sqrt{105}}{\pi^2}\cdot\frac{(N^2e)^N}{N^{20}}$$

| $K$ | $\alpha$ | $N$ | Estimated number | Exact number | Exact/estimate |
|---|---|---|---|---|---|
| 3 | 3 | 7 | 6.9679 | 161 | 23.1 |
| 3 | 3 | 8 | 5.4403*10^2 | 17218 | 31.6 |
| 3 | 3 | 9 | 7.4781*10^4 | 363949 | 4.87 |

These results are quite poor indeed, but gradual improvements can be expected for increasing values of $N$, which unfortunately cannot be verified because no more exact values are known.

At least, such a gradual improvement seems to occur in the case of trimagic series for *squares*, for which more exact results are available (see http://www.multimagie.com/English/Series.htm):

$$\mathcal{N}_{2,g}^{(3)}(N) \approx \frac{720\sqrt{105}}{\pi^2\sqrt{e}}\cdot\frac{(Ne)^N}{N^{14}}$$

| $K$ | $\alpha$ | $N$ | Estimated number | Exact number | Exact/estimate |
|---|---|---|---|---|---|
| 3 | 2 | 7 | 0.6037 | 0 | 0 |
| 3 | 2 | 8 | 5.1558 | 121 | 23.5 |
| 3 | 2 | 9 | 6.2218*10^1 | 126 | 2.03 |
| 3 | 2 | 11 | 2.0396*10^4 | 31187 | 1.53 |
| 3 | 2 | 12 | 5.1245*10^5 | 2226896 | 4.35 |
| 3 | 2 | 13 | 1.5430*10^7 | 17265701 | 1.12 |
| 3 | 2 | 15 | 2.2233*10^10 | 69303997733 | 3.12 |
| 3 | 2 | 16 | 1.0314*10^12 | 1683487116508 | 1.63 |
| 3 | 2 | 17 | 5.3806*10^13 | 112205432382966 | 2.09 |

As can be seen from this last table, the oscillations seem to become smaller for increasing values of $N$ (but will never disappear completely), and one is inclined to expect the same thing to happen for cubes and for hypercubes of higher dimensions as well.

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