60^6 is the smallest possible magic product of 6x6 pandiagonal multiplicative magic squares

by Oscar Lanzi, May 2015

Herein is a proof that the smallest possible magic product in a 6^{th}-order, pandiagonal, multiplicative magic square consisting of distinct positive whole numbers (henceforth referred to as a PMMS-6) is 60^6. This is the product first discovered by Radko Nachev in 2012.

The proof begins with two lemmas. Lemma 1 establishes that the PMMS-6 with a minimum product has two or three distinct prime factors, no more or less. Lemma 2 asserts that 6th-order pandiagonal additive squares have a uniform sum for every set of four corners of a 4x4 block. These lemmas will be used to set up pigeonhole-type arguments that eliminate all possible lower products than Nachev's.

The main proof begins with the case of three distinct prime factors (which would be 2, 3 and 5). Then the case of two prime factors (2 and 3) is considered. The latter case requires two separate arguments depending on whether the minimum magic product has (at least) 12 powers of 3 or only 6. This is what is ultimately found:

1) With three prime factors the minimum product matches the known one of 60^6.

2) With two prime factors and a product having at least 12144^6 powers of 3, the minimum product must be at least 72^6.

3) With two prime factors and a product with only six powers of 3, the minimum product must be at least 96^6.

<u>Lemma 1: The minimum product PMMS-6 will have either two or three distinct prime factors.</u>

This is proven indirectly. If the PMMS-6 has just one distinct prime factor, then the minimum product case occurs when that factor is 2. But, to accommodate 36 distinct nonconsecutive powers of 2 (consecutive powers of 2 are not allowed because the exponents have to form an additive pandiagonal square, which in sixth order cannot consist of consecutive elements), the maximum entry must be at least 2^{36} and the magic product must be at least $2^{108} = 262{,}144^6$.

Now suppose that the PMMS-6 has four or more distinct prime factors. From Morgenstern's proof in 2007, the magic product of the PMMS-6 must be a sixth power. With four or more distinct prime factors such a number must be at least $2^6 3^6 5^6 7^6 = 210^6 > 60^6$. Thus the minimum product can be realized neither with four or more distinct prime factors, nor with just one. That leaves either two or three distinct prime factors for a PMMS-6 with the minimum product.

Lemma 2: In a 6th-order, additive pandiagonal magic square, the sum of the corners of any 4x4 block equals 2/3 of the magic sum.

Suppose that the elements of a 6th-order, additive pandiagonal magic square are labeled as follows:

A_{11}	B_{11}	C_{11}	A_{12}	B_{12}	C_{12}
D_{11}	E_{11}	F_{11}	D_{12}	E_{12}	F_{12}
G_{11}	H_{11}	I_{11}	G_{12}	H_{12}	I_{12}
A_{21}	B_{21}	C_{21}	A_{22}	B_{22}	C_{22}
D_{21}	E_{21}	F_{21}	D_{22}	E_{22}	F_{22}
G_{21}	H_{21}	I_{21}	G_{22}	H_{22}	I_{22}

Then each set of elements sharing a common letter (A, B, C, …, I) constitute the corners of a 4x4 block. Although there are 36 such blocks when wraparound is allowed, the wrapping process also causes groups of four blocks to share a common set of corners. So the nine sets of corners defined above by the letters A through I are all such sets in the square.

Let T(A) be the total of all the A squares, T(B) be the total of all the B squares, and so on through letter I. Collectively the A, B, and C elements match with the first and fourth rows; D, E and F correspond to the second and fifth rows; and G, H and I correspond to rows 3 and 6. Then we have:

$$T(A)+T(B)+T(C)=T(D)+T(E)+T(F)=T(G)+T(H)+T(I)=2S \quad (1)$$

S is the magic sum of the square. Similarly, groups of letters are aligned along the columns and diagonals giving rise to additional equations:

$$T(A)+T(D)+T(G)=T(B)+T(E)+T(H)=T(C)+T(F)+T(I)=2S \quad (2)$$

$$T(A)+T(E)+T(I)=T(B)+T(F)+T(G)=T(C)+T(D)+T(H)=2S \quad (3)$$

$$T(A)+T(C)+T(H)=T(B)+T(D)+T(I)=T(C)+T(E)+T(G)=2S \quad (4)$$

Now suppose that the four-corner sums are arranged into their own 3x3 matrix:

T(A)	T(B)	T(C)
T(D)	T(E)	T(F)
T(G)	T(H)	T(I)

Then Eqs. (1)-(4) require this array to be a pandiagonal additive square, but for a 3x3 array this is possible only if all elements are identical. Therefore all the sets of corners of 4x4 blocks have a common sum. If this is called U, then any of Eqs. (1-4) reduces to 3U = 2S, so U = 2S/3.

<u>Proof of the Minimum Product for Three Distinct Prime Factors</u>

With three distinct prime factors, the minimum product is of curse obtained when the prime factors are 2, 3 and 5. Then, let M be the PMMS-6. Label the element in row I and column j of M as M(I, j). Then we have:

$$M(i,j) = 2^{E2(i,j)}3^{E3(i,j)}5^{E5(i,j)} \quad (5)$$

E2, E3, and E5 are exponent matrices associated with the prime factors 2, 3, and 5 respectively. Each consists of nonnegative integers and is additive-pandiagonal. Let their respective magic sums be S(2), S(3) and S(5), all of which will be multiples of 6. Then the magic product of M is given in terms of the exponent matrices by:

$$P(M) = 2^{S(2)}3^{S(3)}5^{S(5)} \quad (6)$$

Next define a T matrix as the sum of the exponent matrices:

$$T(i,j) = S2(i,j) + S3(i,j) + S5(i,j) \quad (7)$$

$$S(T) = S(E2) + S(E3) + S(E5) \quad (8)$$

With Eq. (5), each element M(I, j) of the multiplicative square is associated with an ordered triple of exponents (E2(I, j), E3(I, j), E5(I, j)). For all elements of M to be distinct, the ordered triples must be distinct. This places limits on how small the T matrix can be. Thirty-six ordered triples are required, but:

- Only 35 have T(I, j) < 5, so one must have T(I, j) ≥ 5
- Only 20 have T(I, j) < 4, so 15 more (not counting the one noted just above) must have T(I, j) ≥ 4
- Only 10 have T(I, j) < 3, so 10 more must have T(I, j) ≥ 3
- Only four have T(I, j) < 2, so six more must have T(I, j) ≥ 2
- Only one has T(I, j) < 1, so three more must have T(I, j) ≥ 1

Then the total of all 36 elements in the T matrix must satisfy:

$$6S(T) = 6(S(E2) + S(E3) + S(E5)) \geq 5 + 15(4) + 10(3) + 6(2) + 3(1) = 110 \quad (9)$$

S(2), S(3), and S(5) must each be a positive multiple of 6, and S(2) = S(3) = S(5)= 6 would give 6S(t) = 108 which is contradicted by (9). So the smallest possible magic product P(M) with three distinct prime factors as given by Eq. (6) is $2^{12}3^{6}5^{6} = 60^{6}$.

Proof of the Minimum Product for Two Distinct Prime Factors

By Lemma 1, the just completed proof the case of three distinct prime factors implies that a product less than 60^{6} must come from a matrix with two distinct prime factors, which would then be 2 and 3. Analogously to the three-factor case, we define E2 and E3 and exponent matrices corresponding to these prime factors, each having nonnegative integer entries and positive pandiagonal sums which are divisible by 6. The analogues to Eqs. (5-8) above are then:

$$M(i,j) = 2^{E2(i,j)}3^{E3(i,j)} \quad (10)$$

$$P(M) = 2^{S(2)}3^{S(3)} \quad (11)$$

$$T(i,j) = S2(i,j) + S3(i,j) \quad (12)$$

$$S(T) = S(E2) + S(E3) \quad (13)$$

For the minimum product S(2) must be greater than or equal to S(3), otherwise a lower product would be obtained by interchanging the exponent matrices. Then, if a product below Nachev's is possible, S(E3) cannot be as large as 18 because $2^{18}3^{18}=216^{6} > 60^{6}$. S(E3) must be either 6 or 12.

If S(3) = 12, then by Lemma 3 each element E3(I, j) in the E3 matrix may be as large as 8 (the sum of the 4x4 block corners), and the same holds true for the E2 elements because S(E2) ≥ 12. Given this allowance, nevertheless it is true that:

- Only 28 distinct ordered pairs (E2(i, j), E3(I, j)) have T(I, j) < 7, so eight must have T(I, j) ≥ 7
- Only 21 have T(I, j) < 6, so seven more must have T(I, j) ≥ 6
- Only 15 have T(I, j) < 5, so seven more must have T(I, j) ≥ 5
- Only 10 have T(I, j) < 4, so seven more must have T(I, j) ≥ 4
- Only six have T(I, j) < 3, so seven more must have T(I, j) ≥ 3
- Only three have T(I, j) < 2, so seven more must have T(I, j) ≥ 2
- Only one has T(I, j) < 1, so seven more must have T(I, j) ≥ 1

Then, in the same manner as Eq. (9), the following is required for all element of M to be distinct:

$$6S(T) = 6\big(S(E2) + S(E3)\big) \geq 8(7) + 7(6) + 6(5) + 5(4) + 4(3) + 3(2) + 2(1) = 168 \quad (14)$$

With S(E3) = 12 and S(E2) required to be a multiple of 6, S(E2) has to be at least 18 and thus P(M) has to be at least $2^{18}3^{12} = 72^{6}$.

What remains is the case where S(E3) = 6. Here, Eq. (14) does not provide a sufficiently stringent bound because $2^{24}3^6 = 48^6$. But Eq. (14) is also a weak bound because it does not account for the fact that, with S(E3) = 6, the individual elements E3(I, j) cannot be greater than 4. Lemma 2 imposes that constraint.

So, a different kind of pigeonhole argument is developed to complete the proof the case of two distinct prime factors with S(E3) = 6. Let n(k) be the number of elements in the E3 matrix equal to k for k = 0, 1, 2, 3, or 4. Then each of the n(0) positions in the E3 matrix where E3(I, j) = 0 must correspond to a different value in the corresponding position of the E2 matrix. At a minimum, the E2 elements in these positions must have a sum at least as large as the sum of the first n(0) nonnegative integers. A similar constraint holds for E3(I, j) = 1, 2, 3 or 4. Then the total of all 36 elements in the E2 matrix must satisfy:

$$6S(E2) \geq (\frac{1}{2}) \sum_{k=0}^{4} n(k)(n(k)-1) \quad (15)$$

Two constraints are now placed on n(k). First, there must be 36 total elements in the E3 matrix; second, they must add up to 36 for the case being studied here:

$$\sum_{k=0}^{4} n(k) = 36 \quad (16)$$

$$\sum_{k=0}^{4} kn(k) = 36 \quad (17)$$

There are many solutions to Eqs. (16) and (17) in which all the n(k) are nonnegative. However, when they are substituted into (15) the lower bound obtained on the right side of that equation is always greater than or equal to 177. Thus S(E2) is at least 30 when there are two distinct prime factors and S(E3) = 6. This gives $P(M) \geq 96^6$ for this case.

So, the requirement that a PMMS-6 have positive, distinct whole number elements leaves no possibility of a product less than 60^6 with either two or three prime factors, and Lemma 1 rules out such a product for any other number of prime factors. Therefore 60^6 is an optimized product for 6^{th}-order, pandiagonal, multiplicative squares containing distinct, positive whole numbers.