Earlier in 2013, Lee Morgenstern gave a number of proofs for tetramagic, pentamagic and hexamagic series or squares. These proofs based on properties modulo M ($M = 2$, 3, 5, 7) may be expressed as more general links between the "magic sums" $S_k = \frac{1}{n}\sum_{m=1}^{n^2} m^k$ and the number of entries according to their remainders modulo M.

Nota. Series of order $n = 4m + 2$ cannot be trimagic, because S_1 is odd and S_3 is even.

$M = 2$ (odd/even entries)

Let n_1 the number of odd entries.

• Tetramagic series

$n_1 = S_4 \pmod{16}$

Special cases

a/ If $n_1 = 1$, $S_4 - 4S_3 - 4S_2 + 16S_1 \pmod{128} \in \{9, 49, 105, 113\}$
Example: $n = 17$, 25, 49 or 57, $S4 - 4S_3 - 4S_2 + 16S_1 = 73 \pmod{128}$, so n_1 cannot be 1.

b/ Suppose there is no even entry ($n_1 = n$). Then $2S_2 - S_4 = n \pmod{64}$ which may be a contradiction.
Example: $n = 16k + 1$, $2S_2 - S_4 = 1 \pmod{64}$, so n_1 cannot be n when k is not a multiple of 4.
$n = 16k - 1$, $2S_2 - S_4 = 63 \pmod{64}$.

• Pentamagic series

$n_1 = S_4 - 6S_3 + 6S_5 \pmod{32}$.
Example: $n = 19$, $S_4 - 6S_3 + 6S_5 = 23 \pmod{32}$; $n_1 \geq 23 > 19 = n$.
$n = 23$, $S_4 - 6S_3 + 6S_5 = 31 \pmod{32}$; $n_1 \geq 31 > 23 = n$.
$n = 17$ or 25, $n_1 = 1 \pmod{32}$; in contradiction with the tetramagic condition (special case a/).
$n = 31$, 39, 47, 55, 63: $n_1 = 31 \pmod{32}$, so n_1 cannot be 15 (nor 47 if $n \geq 47$)
$n = 33$, 41, 49, 57, 65: $n_1 = 1 \pmod{32}$, so n_1 cannot be 17 (nor 49 if $n \geq 49$)

• Hexamagic series

$n_1 = 2S_6 - 5S_4 + 4S_2 \pmod{64}$.
Example: $n = 36$, $2S_6 - 5S_4 + 4S_2 = 50 \pmod{64}$; $n_1 \geq 50 > 36 = n$.
Similar impossibility for $n = 19$, 20, 21, 23, 25, 28, 29, 31, 37, 44, 47, 52, 60.

$\underline{M = 3}$

Let n_k ($k = 1, 2$) the number of entries with remainder k modulo 3.

• Tetramagic series

By Lee's Modulo 9 Tetramagic Lemma

$n_1 = 4S_2 - 4S_3 + S_4 \pmod 9$, $n_2 = 4S_2 + 4S_3 + S_4 \pmod 9$.
Example: $n = 20$, $n_1 = 4 \pmod 9$, $n_2 = 8 \pmod 9$.

• Pentamagic series

$n_1 - n_2 = 7S_3 - 6S_5 \pmod{27}$

• Hexamagic series

$n_1 = 20S_6 - 3S_5 - 10S_3 - 6S_2 \pmod{27}$,
$n_2 = 20S_6 + 3S_5 + 10S_3 - 6S_2 \pmod{27}$.
Example: $n = 24$, $n_1 = n_2 = 26 \pmod{27}$; $n_1 + n_2 \geq 52 > 24 = n$.
Similar impossibility ($n_1 + n_2 > n$) for $n = 17, 19, 21, 24, 28, 31, 32, 35, 37, 39, 43, 47, 48$.

$\underline{M = 5}$

Let n_k ($k = 1, 2, 3, 4$) the number of entries with remainder k modulo 5.

• Tetramagic series

By Lee's Modulo 5 Tetramagic Lemma (notice that $S_1 = S_2 = S_3 \pmod 5$).

$n_1 = 2S_1 - S_4 \pmod 5$, $n_2 = n_3 = n_4 = S_1 - S_4 \pmod 5$.
Example: $n = 12$, $n_1 = n_2 = n_3 = n_4 = 2 \pmod 5$.
$n = 16$, $n_1 = 2 \pmod 5$, $n_2 = n_3 = n_4 = 1 \pmod 5$.

• Pentamagic series

$S_5 = S_1 \pmod 5$ by Fermat's theorem.

$n_1 + 7n_2 - 7n_3 - n_4 = S_5 \pmod{25}$.

Example: $n = 16$, $n_1 + 7n_2 - 7n_3 - n_4 = 11 \pmod{25}$

• Hexamagic series

$n_1 - n_2 - n_3 + n_4 = 12S_6 - 11S_2 \pmod{25}$.

Example: $n = 16$, $n_1 - n_2 - n_3 + n_4 = 11 \pmod{25}$,
$n_1 = 12$, $n_2 = n_3 = n_4 = 1$. [1]

[1] Lee's proof for hexamagic series of order 16 takes advantage of this distribution for studying the entries with remainder 0, 2, 3, 4 modulo 5. The result is that some of these entries must be $> 256 = n^2$, which proves impossibility. This approach cannot be put as a general formula.

$M = 7$, Hexamagic series

Let n_k $(0 \leq k \leq 6)$ the number of entries with remainder k modulo 7.

By Lee's Modulo 7 Hexamagic Lemma
(notice that $S_4 = 2S_1 + 2S_2 - 3S_3 \pmod 7$, $S_5 = 3S_1 - 2S_3 \pmod 7$).

$n_1 = S_1 - 3S_2 - 3S_3 - S_6 \pmod 7$,

$n_2 = 3S_1 - 3S_2 + S_3 - S_6 \pmod 7$,

$n_3 = n_4 = 3S_1 - S_2 - S_3 - S_6 \pmod 7$,

$n_5 = n_6 = 2S_1 - 3S_2 + 2S_3 - S_6 \pmod 7$.

Example: $n = 31$, $n_1 = n_2 = 4 \pmod 7$,
$n_3 = n_4 = n_5 = n_6 = 6 \pmod 7$. $\sum_k n_k \geq 32 > 31 = n$.
Similar impossibility for $n = 19$, 20, 23.

Unfortunately, the modulus $M = 7$ brings no new proof of impossibility; these hexamagic series were already proved impossible with $M = 2$ or $M = 3$.