# The Lost Theorem

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"Almost the last word has been said on this subject"
— H. E. Dudeney on magic squares [1]

A magic square, as all the world knows, is a square array of numbers whose sum in any row, column, or main diagonal is the same. So-called "normal" squares are ones in which the numbers used are 1,2,3, and so on, but other numbers may be used. Squares using repeated entries are deemed trivial. We say that a square of size $N \times N$ is of order $N$. Clearly, magic squares of order-1 lack glamour, while a moment's thought shows that a square of order 2 cannot be realized using distinct entries. The smallest magic squares of any interest are thus of order-3.

Writing in *Quantum* recently [2], Martin Gardner offered $100 to anyone able to produce a 3×3 magic square composed of any nine distinct *square* numbers. So far nobody has produced a solution, or proof of its impossibility, although a near miss I discovered is the following:

| | | |
|---|---|---|
| $127^2$ | $46^2$ | $58^2$ |
| $2^2$ | $113^2$ | $94^2$ |
| $74^2$ | $82^2$ | $97^2$ |

a specimen whose rows, columns, and just *one* of the two main diagonals sum to the same number, itself a square: $147^2$. It was while tinkering in connection with this problem that I was startled to discover an elementary correspondence between 3×3 magic squares and *parallelograms*. The reason for my surprise is worth explaining.

Magic squares have been a special hobby of mine for over twenty years; the literature on the topic, much of which I have collected, is extensive. As already noted, 3×3 magic squares are the smallest and hence simplest types, for which reason they are the earliest to appear in history, as well as being the most thoroughly investigated squares of all. Innumerable books and articles on magic squares begin with a discussion of 3×3 types, the properties of which have long been regarded as completely understood. Writing in the well known *Mathematical Recreations* published in 1930, for instance, Maurice Kraitchik begins by saying that, "The theory of the squares of the third order is simple and complete ..", and then goes on to present that theory in just two pages of text.

Yet for all its extreme simplicity, the elementary correspondence with parallelograms that I had stumbled upon while working on Gardner's problem, has, to the best of my knowledge, never previously been identified. I feel sure that many readers will share my incredulity on inspecting the theorem below. They may agree with me that the correlation with parallelograms it describes is so basic that it deserves to be regarded as *the* fundamental theorem of order-3 magic squares, and the very first thing that any newcomer to the subject should learn. How then could such a theorem escape the attention of every researcher in the field from ancient times down to the present day?

An explanation lies in the orthodox focus on magic squares using natural numbers. Once our attention broadens to include squares that use *complex* numbers, the familiar integer types become only a special case, preoccupation with which has obscured the wider picture. Moving beyond this narrow view, we step into a realm of greater clarity and harmony. And at the very center of that realm we shall find an undiscovered prize, the atomic magic square.

*Standard Theory*

What makes a 3×3 square magic?
The well-known algebraic formula due to Édouard Lucas describes the structure of every magic square of order-3:

| | | |
|---|---|---|
| $c-b$ | $c+a+b$ | $c-a$ |
| $c-a+b$ | $c$ | $c+a-b$ |
| $c+a$ | $c-a-b$ | $c+b$ |

Lucas's formula conveys much of the essential information in a single swoop. In particular, we can see at a glance that the constant total, which is 3*c*, is equal to three times the center number, while a closer look shows that whatever the nine numbers used in the square, they must always include eight 3-term arithmetic progressions, namely:

*1 : c+a, c, c-a,*
*2 : c+b, c, c-b,*
*3 : c+a+b, c, c-a-b,*
*4 : c+a-b, c, c-a+b,*
*5 : c+a-b, c+a , c+a+b*
*6 : c-a+b, c-a, c-a-b,*
*7 : c+a+b, c+b, c-a+b,*
*8 : c+a-b, c-b, c-a-b,*

The identification of these arithmetic triads is a recurrent feature in discussions of order-3 theory, a point we shall return to later, although it is rare to find an explicit list of all eight.

Of course, just as any magic square can be rotated and reflected to result in 8 trivially distinct squares that are deemed equivalent, so there are 8 trivially distinct rotations and reflections of the formula, all of them isomorphic to each other, and again comprising one equivalence class.

So much for a bird's eye view of the theory of order 3 magic squares as it is met within the literature. Let us now turn our attention elsewhere.

*Complex Squares*.

Consider Figure 1, which depicts an arbitrary parallellogram, *PQRS*, centered at some arbitrary point, *M*, on the Euclidean plane, with axes *X* and *Y*. Point *O* is the origin of the plane. The corner points, *P, Q, R*, and *S*, together with *T, U, V*, and *W*, the midpoints of the sides of the parallelogram, as well as the center, *M*, can thus each be identified with vectors or complex numbers of form $x + y\mathbf{i}$, in which *x* and *y* are the real number coordinates of each point, and $\mathbf{i} = \sqrt{-1}$ .

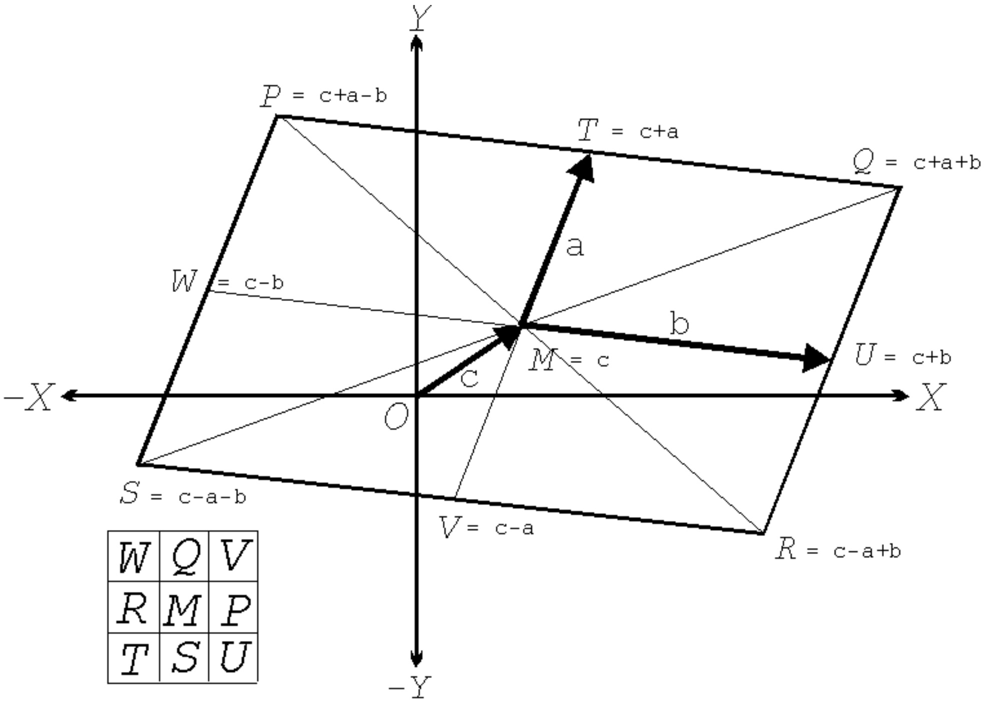


Figure 1

Equally, the lines connecting these points may themselves be interpreted as vectors, three of which are identified in the Figure as: $\overrightarrow{MT}$ = **a**, $\overrightarrow{MU}$ = **b**, and $\overrightarrow{OM}$ = **c**. Note that given any three particular complex values for **a**, **b** and **c**, we could immediately proceed to construct the corresponding parallelogram.

Now by the law for the addition of vectors, the point or complex number, *T* (which is the vector *OT*), is the resultant of the two vectors **c** and **a**, or **c** + **a**. And likewise, it takes but a glance to identify the remaining points on *PQRS* in terms of the vectors, **a**, **b**, and **c**, as indicated in the Figure: *P* = **c** + **a** - **b**, *Q* = **c** + **a** + **b**, *R* = **c** - **a** + **b**, *S* = **c** - **a** - **b**, *U* = **c** + **b**, *V* = **c** - **a**, *W* = **c** - **b**. and *M* = **c**.

Looking next at the 3×3 square shown below left in Figure 1, observe what happens when a new square is created by replacing *P*,*Q*,.. with their corresponding expressions in terms of **a**, **b** and **c**:

| | | |
|---|---|---|
| **c − b** | **c + a + b** | **c − a** |
| **c − a + b** | **c** | **c + a − b** |
| **c + a** | **c − a − b** | **c + b** |

The outcome is nothing less than a reappearance of Lucas's formula for 3×3 magic squares.

The implication is as obvious as it is surprising: given any particular parallelogram on the Euclidean plane, and then transcribing the complex numbers corresponding to its four corners, four edge midpoints, and center, into a 3×3 matrix, in the same way as above, the resulting square will *always* be magic. Or alternatively, starting with any 3×3 magic square that uses complex number entries, we will find that they define nine points on the Euclidean plane that coincide with the four corners, four edge midpoints, and center of a parallelogram

In summary, we have:

**Theorem**. *To every parallelogram drawn on the plane there corresponds a unique equivalence class of 8 complex 3×3 magic squares, and for every equivalence class of 8 complex 3×3 magic squares there corresponds a unique parallellogram on the plane*.

Or in a nutshell: rotations and reflections disregarded, every parallelogram defines a unique 3×3 magic square, and vice versa.

In this light it is interesting to recall the eight arithmetic progressions previously identified in every 3×3 magic square. For just as arithmetic progressions of real numbers correspond to equidistant points along the real number line, so arithmetic progressions of complex numbers correspond to equidistant *colinear* points on the plane. See then how the eight progressions listed earlier precisely correlate with the eight sets of 3 colinear points lying along the four edges and four bisectors of the parallelogram in Figure 1:

*1 : c+a, c, c-a = T,M,V*
*2 : c+b, c, c-b = U,M,W*
*3 : c+a+b, c, c-a-b = Q,M,S*
*4 : c+a-b, c, c-a+b = P,M,R*
*5 : c+a-b, c+a, c+a+b = P,T,Q*
*6 : c-a+b, c-a, c-a-b = R,V,S*
*7 : c+a+b, c+b, c-a+b = Q,U,R*
*8 : c+a-b, c-b, c-a-b = P,W,S.*

In the magic square literature to date, discussion of theory never gets further than *identifying* these progressions; now at last we can see how the geometry of the parallellogram *explains* their presence.

From our new perspective we can see also how the corellation with parallelograms has escaped previous notice. Traditionally magic squares have used integers, which are entries without imaginary component. The parallelograms corresponding to these squares are thus collapsed or *degenerate* specimens of zero area, making their presence undetectable. In fact, a closer look at one such parallelogram will prove instructive, as well as preparing us for an unexpected development: the discovery of a lost archetype, the primordial magic square.

*A Flaw in the Crystal*

I suppose that, until now, the most obvious candidate for the title of archetypal magic square would have been the Chinese *Lo shu*, the simplest, oldest, and most well known square of all. Its magic sum is 15:

| | | |
|---|---|---|
| 2 | 9 | 4 |
| 7 | 5 | 3 |
| 6 | 1 | 8 |

Legend has it that the *Lo shu* was first espied by King Yü on the back of a sacred turtle that emerged from the river Lo in the 23rd century BC. In fact, historical references to the square date from the 4th century BC, while Cammann has argued that it played a major part in Chinese philosophical and religious thought for centuries afterward [3]. In the West, the *Lo shu* has long been held up as a paradigm, or "one of the most elegant patterns in the history of combinatorial number theory," as Martin Gardner has written. Nevertheless, taking a lens to this ancient gem, we can discover an interesting irregularity in its crystal lattice.

Consider the *Lo shu*'s flattened parallelogram, which is that segment of the real number line between 1 and 9, along which lie its four corners, four edge midpoints, and center, occupying nine equidistant points. Recalling Figure 1, the relation between these points and their position in the magic square can be diagrammed as follows:

```
    3  6  9          2  9  4
  2  5  8     ⇔     7  5  3
1  4  7              6  1  8

   Parallelogram      Magic Square
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The distance between the parallelogram's corners at points 1 and 3 (or 7 and 9 ) is thus 2 units, while the distance between those at points 3 and 9 (or 1 and 7) is 6 units; a ratio in side lengths of 1:3. The remarkable fact is thus that the *Lo shu* parallelogram is not equilateral, which is a bit disappointing for a pattern whose famous *symmetry* has won acclaim down the ages, from the banks of the river Lo to the pages of *Scientific American*. It is beginning to look as if that turtle was not quite as sacred as King Yü had imagined.

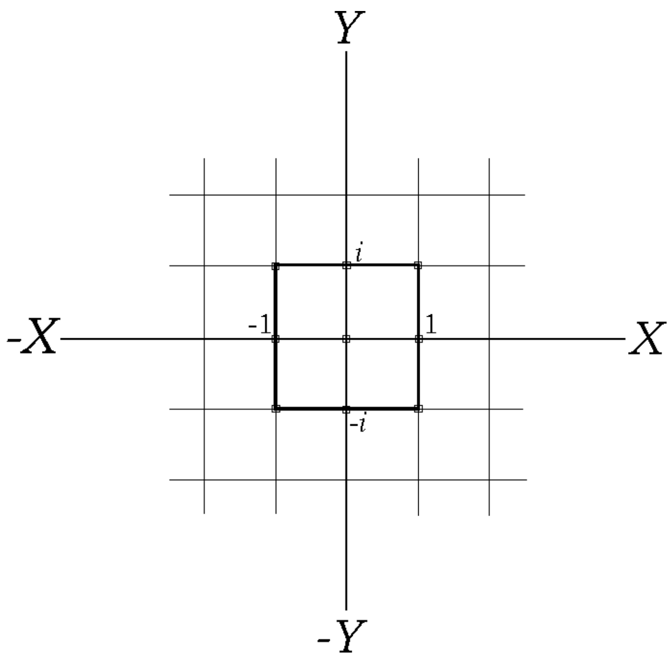
However, as a prisoner in Flatland, the *Lo shu* is doomed to this imbalance: squash any *equilateral* parallelogram, and two pairs of edge midpoints and two corners will *coincide*, forcing repeated entries in the associated magic square. In other words, unless it is trivial, an asymmetrical parallelogram must accompany *every* non-complex magic square, the one on the (mock?) turtle included.

Where then is the magic square with the symmetrical parallelogram that King Yü was denied?

*The Atomic Square*

Of course, the most symmetrical case of all is an equilateral parallelogram that is equiangular as well: the *square*.

A magic square whose associated parallelogram is again a square; the idea is at once compelling. But what kind of a magic square would that be? To find out, all we have to do is draw a square on the plane, read off the complex values of its four corners, etc, and then write these into a 3×3 matrix in the usual way. Simplest of all is this canonical or atomic case:




It is a square centered on the origin of the plane, such that its four corners and four edge midpoints coincide with the 8 complex integers immediately surrounding the origin. The magic square corresponding to this geometric square is consequently an atomic paradigm of its kind too: it is the smallest, most perfectly symmetrical magic square, composed of the nine smallest gaussian integers:

| | | |
|---|---|---|
| $-\mathbf{i}$ | $1+\mathbf{i}$ | $-1$ |
| $-1+\mathbf{i}$ | $0$ | $1-\mathbf{i}$ |
| $1$ | $-1-\mathbf{i}$ | $\mathbf{i}$ |

The elegance of this flawless prism is beyond compare. The two main diagonals and two central orthogonals are like four balanced beams pivoted on the center number, the integer at the end of each beam offset by its opposing negative image, an equipoise reflected in the magic sum of zero. Rewriting the square in the form of vectors as ordered pairs, [a,b], its structural harmony reappears in the shape of palindromic rows and antipalindromic columns, a quality that is better highlighted when $\overline{1}$ replaces $-1$, and the commas and brackets are discarded:

| | | |
|---|---|---|
| $0\bar{1}$ | 11 | $\bar{1}0$ |
| $\bar{1}1$ | 00 | $1\bar{1}$ |
| 10 | $\overline{11}$ | 01 |

Analysing the square in terms of Lucas's formula, we find that the variables *a* and *b* have here taken on the values of 1 and **i**, the real and imaginary forms of unity, while *c* is equal to zero. Could anything be more natural, or poetic?

My interest in magic squares began a couple of decades ago when I first encountered the *Lo shu*. I recall my delight in exploring its symmetries, but I also recall my disquiet in detecting a strange lopsidedness. In Lucas's formula, the variables *a* and *b* appear in two patterns that are perfect mirror images. In the *Lo shu*, however, $a = 1$, while $b = 3$, a numerical imbalance that clashed with the symmetry of the patterns. Attempts to construct a square in which $a = b$, or $a = -b$, wouldn't work either, because the result is then trivial. Yet a craving for symmetry is what makes the mathemagical mind tick. Down the years this unease has continued to quietly smoulder, until recent events brought the parallelogram theorem to light, and with it a sudden resolution of the mystery in the shape of the atomic square, whose symmetry is without flaw. It is a relief; I look forward to sleeping at nights once again.

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References

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[2] M. Gardner, The Magic of 3×3, Quantum, January/February 1996, pp. 24-6
[3] S. Cammann, The Magic Square of Three in Old Chinese Philosophy and Religion, History of Religions I (1961), pp. 37-80