The Magic Hourglass of Squares related to the Gaussian Integers

Christian Woll

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The contents of this paper will consider the magic hourglass of squares. This mathematical object is composed of 7 squares arranged in a 3x3 grid such that the entries of the top row, bottom row, middle column, and both diagonals each sum to the same total. It is not currently known if one exists. If one did, it would look like this

$$\begin{bmatrix} A^2 & B^2 & C^2 \\ \text{---} & E^2 & \text{---} \\ G^2 & H^2 & I^2 \end{bmatrix}$$

Equivalently this means for $A, B, C, E, G, H, I \in \mathbb{Z}$ the following 5 sums are equal to the same total, say T

$$A^2 + B^2 + C^2 = T$$

$$G^2 + H^2 + I^2 = T$$

$$A^2 + E^2 + I^2 = T$$

$$B^2 + E^2 + H^2 = T$$

$$C^2 + E^2 + G^2 = T$$

A potential method of construction using the Gaussian integers will be presented first.

Theorem 1. If there exists 3 distinct Gaussian integers $z_1, z_2, z_3 \in \mathbb{Z}[i] - \mathbb{R}$ such that

$$1) \quad ||z_1|| = ||z_2|| = ||z_3||$$

$$2) \quad z_1^4 + z_2^4 + z_3^4 \in \mathbb{R}$$

then there exists a magic hourglass of distinct squares of integers. Specifically, the 5 aforementioned sums will be satisfied by choosing

$$A = \text{Im}[(1+i)z_1^2] \quad B = \text{Im}[(1+i)z_2^2] \quad C = \text{Im}[(1+i)z_3^2]$$

$$E = ||z_1||^2 = ||z_2||^2 = ||z_3||^2$$

$$G = \text{Re}[(1+i)z_3^2] \quad H = \text{Re}[(1+i)z_2^2] \quad I = \text{Re}[(1+i)z_1^2]$$

Proof.

Lemma 1.1. $2(||z||^2)^2 = \mathrm{Re}[(1+i)z^2]^2 + \mathrm{Im}[(1+i)z^2]^2$
For $z \in \mathbb{Z}[i]$

$$2(||z||^2)^2$$
$$||$$
$$||1+i||^2 \cdot ||z||^4$$
$$||$$
$$||(1+i)z^2||^2$$
$$||$$
$$\mathrm{Re}[(1+i)z^2]^2 + \mathrm{Im}[(1+i)z^2]^2$$

Now substituting z_1,z_2, and z_3 for z respectively gives the equations

$$A^2 + I^2 = 2E^2 \tag{1}$$
$$B^2 + H^2 = 2E^2 \tag{2}$$
$$C^2 + G^2 = 2E^2 \tag{3}$$

Lemma 1.2. $2Im[z^4] = \mathrm{Im}[(1+i)z^2]^2 - \mathrm{Re}[(1+i)z^2]^2$
For $z \in \mathbb{Z}[i]$

$$2Im[z^4]$$
$$||$$
$$-Re[z^4 \cdot 2i]$$
$$||$$
$$-Re[(z^2(1+i))^2]$$
$$||$$
$$\mathrm{Im}[(1+i)z^2]^2 - \mathrm{Re}[(1+i)z^2]^2$$

Again substituting z_1,z_2, and z_3 for z respectively gives the equations

$$A^2 - I^2 = 2Im[z^4] \tag{4}$$
$$B^2 - H^2 = 2Im[z^4] \tag{5}$$
$$C^2 - G^2 = 2Im[z^4] \tag{6}$$

These equalities are now made use of as follows

$$z_1^4 + z_2^4 + z_3^4 \in \mathbb{R} \;\Leftrightarrow\; \mathrm{Im}[z_1^4] + \mathrm{Im}[z_2^4] + \mathrm{Im}[z_3^4] = 0$$
$$\Updownarrow$$
$$A^2 - I^2 + B^2 - H^2 + C^2 - G^2 = 0 \tag{7}$$

Adding equations (1), (2), and (3) to equation (7) yields the following

$$2A^2 + 2B^2 + 2C^2 = 6E^2$$

$$\Updownarrow$$

$$A^2 + B^2 + C^2 = 3E^2$$

Similarly, subtraction equations (1), (2), and (3) from equation (7) yields the following

$$-2G^2 - 2H^2 - 2I^2 = -6E^2$$

$$\Updownarrow$$

$$G^2 + H^2 + I^2 = 3E^2$$

Finally adding E^2 to each of equations (1), (2), and (3) respectively yields

$$A^2 + E^2 + I^2 = 3E^2$$

$$B^2 + E^2 + H^2 = 3E^2$$

$$C^2 + E^2 + G^2 = 3E^2$$

The 5 sums of the magic hourglass have been shown to be equal to the same total – namely $T = 3E^2$ – and thus, the magic hourglass of squares has been (hypothetically) constructed.

Unfortunately the converse of **Theorem 1** is too difficult to prove. That is, that the existence of a magic hourglass of squares implies the existence of the aforementioned Gaussian integers: z_1, z_2, and z_3. Although such difficulty is perceived only from the inability of the author to find such a proof. A weaker result will be shown here.

Theorem 2. If there exists a magic hourglass of distinct squares then there exists 3 distinct complex numbers

$$z_1, z_2, z_3 \in \{z\sqrt{n} : z \in \mathbb{Z}[i] - \mathbb{R}, n \in \mathbb{N}\}$$

such that

$$1) \quad ||z_1|| = ||z_2|| = ||z_3||$$

$$2) \quad z_1^4 + z_2^4 + z_3^4 \in \mathbb{R}$$

Proof.

Lemma 2.1. For any magic hourglass having T as its total and E^2 as its middle element: $T = 3E^2$

$$
\begin{array}{c}
3T \\
|| \\
(A^2+E^2+I^2)+(B^2+E^2+H^2)+(C^2+E^2+G^2) \\
|| \\
(A^2+B^2+C^2)+(E^2+E^2+E^2)+(G^2+H^2+I^2) \\
|| \\
2T+3E^2
\end{array}
$$

From which it follows that $T = 3E^2$

Such a result reveals there are 3 arithmetic sequences of squares in any magic hourglass of squares

$$
\begin{array}{rcl}
A^2+E^2+I^2 = 3E^2 & \Leftrightarrow & A^2-E^2 = E^2-I^2 \\
B^2+E^2+H^2 = 3E^2 & \Leftrightarrow & B^2-E^2 = E^2-H^2 \\
C^2+E^2+G^2 = 3E^2 & \Leftrightarrow & C^2-E^2 = E^2-G^2
\end{array}
$$

Lemma 2.2. All arithmetic sequences of squares are parametrized by a corresponding Gaussian integer and a real integer.

Let r^2, s^2, t^2 form an arithmetic sequence such that

$$r^2 - s^2 = s^2 - t^2$$

Because $2 \cdot s \cdot s$ is the sum of two squares (i.e. $r^2 + t^2 = 2s^2$) it follows that s itself is the sum of two squares, say $s = m^2 + n^2$

The Gaussian integer $z \in \mathbb{Z}[i]$ is now introduced. Let $z = m + ni$ so that

$$s = ||z||^2 = m^2 + n^2$$

Next, the values of r and t are found in terms of m and n as follows

$$
\begin{array}{c}
r^2+t^2 \;=\; ||2s^2||^2 \\
|| \\
||1+i||^2 \cdot ||(m+ni)^2||^2 \\
|| \\
||(1+i)(m^2-n^2+2mni)||^2 \\
|| \\
(m^2-2mn-n^2)^2+(m^2+2mn-n^2)^2
\end{array}
$$

In this way, an arithmetic sequence of squares can be parametrized with any integer m and n

$$r = m^2 - 2mn - n^2 \quad s = m^2 + n^2 \quad t = m^2 + 2mn - n^2$$

Although this is not yet the full solution. The full solution is reached by multiplying r, s, and t by a positive integer constant, say l

$$r = l(m^2 - 2mn - n^2) \quad s = l(m^2 + n^2) \quad t = l(m^2 + 2mn - n^2)$$

Note that multiply r, s, and t by l has the same effect as multiplying m and n by $\sqrt{l}$. Thus we say that $z\sqrt{l}$ is the corresponding parametrization of the sequence r^2, s^2, and t^2. More specifically

$$r = l \cdot \text{Re}[(1+i)z^2] \quad s = l \cdot ||z||^2 \quad t = l \cdot \text{Im}[(1+i)z^2]$$

Because there are 3 arithmetic sequences of squares in a magic hourglass of squares, we may now state using **Lemma 2.2** that there exists $n_1, n_2, n_3 \in \mathbb{N}$ and $x_1, x_2, x_3 \in \mathbb{Z}[i] - \mathbb{R}$ such that

$$A = n_1\text{Im}[(1+i)x_1^2] \quad B = n_2\text{Im}[(1+i)x_2^2] \quad C = n_3\text{Im}[(1+i)x_3^2]$$
$$E = n_1||x_1||^2 = n_2||x_2||^2 = n_3||x_3||^2$$
$$G = n_3\text{Re}[(1+i)x_3^2] \quad H = n_2\text{Re}[(1+i)x_2^2] \quad I = n_1\text{Re}[(1+i)x_1^2]$$

Equivalently, there exists

$$z_1, z_2, z_3 \in \{z\sqrt{n} : z \in \mathbb{Z}[i] - \mathbb{R}, n \in \mathbb{N}\}$$

such that

$$z_1 = x_1\sqrt{n_1} \quad z_2 = x_2\sqrt{n_2} \quad z_3 = x_3\sqrt{n_3}$$

Finally, we show that z_1, z_2, and z_3 have the same norm.

$$\begin{array}{ccccccc} E & = & n_1||x_1||^2 & = & n_2||x_2||^2 & = & n_3||x_3||^2 \\ & & || & & || & & || \\ & & ||x_1\sqrt{n_1}||^2 & = & ||x_2\sqrt{n_2}||^2 & = & ||x_3\sqrt{n_3}||^2 \\ & & || & & || & & || \\ & & ||z_1||^2 & = & ||z_2||^2 & = & ||z_3||^2 \\ & & & & \Downarrow & & \\ & & & & ||z_1|| = ||z_2|| = ||z_3|| & & \end{array}$$

And **Theorem 2** has been proven.