Let \( p \) be a prime of the form \( 8n \pm 3 \). We are going to construct a magic square \( J(p) \) of size \( 2^p \times 2^p \).

We are going to identify integers from 0 to \( 2^p - 1 \) with sequences of their \( p \) binary digits (bits), possibly filled with leading zeros. We refer to positions of bits as from 0-th to \( (p-1) \)-th. It doesn’t really matter whether we put oldest bit last or first as long as we are consistent.

Let for \( 0 \leq i < p \) the sequence \( a_i \) has all bits 0 except \( i \)-th bit, which is 1.

In Mathematica format:

\[ a[0] = \text{Join}\{1\}, \text{Table}[0, \{i, 1, p-1\}] \];
\[ \text{Do}[a[i] = \text{RotateRight}[a[0], i], \{i, 1, p-1\}] ; \]

Let \( a_p \) has 1 on position \( i \) iff \( i \) is quadratic residue mod \( p \), 0 otherwise. We consider \( i = 0 \) to be quadratic residue here.

Let \( a_{p+i} \), where \( 1 \leq i < p \), be \( a_p \) with bits rotated right by \( i \) positions.

\[ a[p] = \text{Ceiling}[\text{Mod}[\text{PowerMod}[\text{Range}[p]-1, (p-1)/2, p]+1, p]/2] ; \]
\[ \text{Do}[a[p+i] = \text{RotateRight}[a[p], i], \{i, 1, p-1\}] ; \]

Let \( b_i = a_{i+1} \) for \( 0 \leq i \leq p-2 \) and \( b_{p-1} = a_0 \).

Let \( b_{p+i} \) be \( a_{p+i-1} \) with all bits reversed, for \( 1 \leq i \leq p-1 \). Let \( b_p \) be \( a_{2p-1} \) with all bits reversed.

\[ \text{Do}[b[i] = a[\text{Mod}[i+1, p]], \{i, 0, p-1\}] ; \]
\[ \text{Do}[b[p+i] = 1 - a[p+\text{Mod}[i+p-1, p]], \{i, 0, p-1\}] ; \]

The table \( J(p) \) has entries

\[ m_{ij} = \sum_{k=0}^{2^p-1} 2^k \cdot (i \circ a_k + j \circ b_k) \mod 2 , \] (♥)

where \( i \circ a_k \) means bitwise multiplication and then adding the products, i.e. counting common occurrences of 1’s in \( i \) and \( a_k \). The sum in parentheses is then taken modulo 2. Indices \( i \) and \( j \) are ranging from 0 to \( 2^p - 1 \).

\[ m = \text{Table}[ \]
\[ \text{Sum}[ \]
\[ 2^k \text{*Mod}[@(\text{Drop}[\text{IntegerDigits}[2^p+i, 2], 1] \ast a[k] + \]
\[ \text{Drop}[\text{IntegerDigits}[2^p+j, 2], 1] \ast b[k]], 2] \], \{k, 0, 2p-1\} ; \]
\[ , \{i, 0, 2^p-1\} ; \{j, 0, 2^p-1\} ; \]
Matrix \emph{J}(p) has consecutive integers from 0 to \(4^p - 1\) as entries if the \(2p \times 2p\) matrix \(X\) whose rows are concatenated \(a_i\) and \(b_i\) has odd determinant.

\[
X = \text{Table}[\text{Join}[a[i], b[i]], \{i, 0, 2p - 1\}];
\]

That is the reason for assuming \(p = 8n \pm 3\).

Let \(c_i\) be bitwise XOR of \(a_i\) and \(b_i\).

We will call a set of \(p\)-bit sequences XOR-\(d\)-independent iff every nonempty subset of at most \(d\) elements has nonzero bitwise XOR of its elements.

If \((a_i)_{0 \leq i \leq 2p}\) is XOR-\(d\)-independent, then square \emph{J}(p) has \(d\)-magic columns.

If \((b_i)_{0 \leq i \leq 2p}\) is XOR-\(d\)-independent, then square \emph{J}(p) has \(d\)-magic rows.

If \((c_i)_{0 \leq i \leq 2p}\) is XOR-\(d\)-independent, then square \emph{J}(p) has both main diagonals \(d\)-magic.

I hope that the above facts are known or can be verified by people deep in the subject. I would hate to go through detailed proof of them. The idea is to replace powers of 2 by variables in (\(\bigtriangledown\)) and to observe that under above XOR-independency conditions, sum of powers up to \(d\)-th of a row, column or diagonal can be expressed without actually looking at particular bits of \(a_i\) and \(b_i\). Note that this sum of powers is a polynomial in \(2p\) variables. Under XOR-independency conditions coefficients of this polynomial are "averaged" the same way, no matter what particular a’s and b’s are.

**Multimagic degree of \emph{J}(p)**

<table>
<thead>
<tr>
<th>(p)</th>
<th>columns</th>
<th>rows</th>
<th>diagonals</th>
<th>square</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>19</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>29</td>
<td>11+</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>37</td>
<td>6+</td>
<td>6+</td>
<td>6+</td>
<td>6+</td>
</tr>
<tr>
<td>43</td>
<td>6+</td>
<td>6+</td>
<td>6+</td>
<td>6+</td>
</tr>
</tbody>
</table>

**Note:** 6+ means I have verified 6-magic (hexamagic) but haven’t tested for 7-magic (heptamagic).

**Checking XOR-independence**

In C: store XOR-sums of \(d\) elements on linked lists. Keep checking whether newly stored XOR-sum is already there. If so, system is not XOR-2d-independent.

If no XOR-sum is repeated, system is XOR-2d-independent provided it has been known to be XOR-(2d-1)-independent.

Keep previously stored XOR-sums of \(d\) elements on linked lists and check them against XOR-sums of \(d+1\) elements. If no sum is repeated, we are sure system is XOR-(2d+1)-independent.

**Remarks**

You can take any integer as \(p\) and any binary vectors as \(a_i\) and \(b_i\) to create your own magic square. But if matrix \(X\) has even determinant, you do not get distinct entries.
If XOR-independence of \((a_i), (b_i)\) and \((c_i)\) is small, multimagic degree of your square is small. You can always present the square by generating 0-th row and 0-th column of the square. The rest is filled as XOR table: \(m_{ij}\) is bitwise XOR of \(m_{0j}\) and \(m_{i0}\).

Files \(ab<p>.txt\) contain \(a_i\) and \(b_i\) in the form of decimal numbers.

In formula (\(\heartsuit\)) you can replace \(2^k\) by ANY numbers and you get multimagic square. You need to put there ANY permutation of powers of 2 to get a square with consecutive integers.

I have verified that \(X\) has odd determinant for \(p = 8n \pm 3\) and \(p < 50\). I have no general proof of that, but I am 99.99999999% sure that is true for all \(p\) of that form.

I feel that multimagic degree of \(J(p)\) tends to \(\infty\) as \(p \to \infty\), but I have no clue how to prove it.

**Using 5magic.exe**

Create file \(ab<p>.txt\) with \(a_i\) and \(b_i\) in decimal form. One number per line, \(a\)'s come first from \(a_0\) to \(a_{2p-1}\), then \(b\)'s. Number \(p\) must be less than 32.

Then run 5magic p

Same applies to 7magic.exe and next programs.

**Decamagic J(29)**

Computations I have performed indicate that \(J(29)\) is 10-magic (decamagic ???). It has size \(2^{29} \times 2^{29}\) or \(536870912 \times 536870912\) and contains integer entries from 0 to \(2^{58} - 1 = 288230376151711743\).

It has 11-magic columns, unlikely 12-magic, but it hasn’t been ruled out at the moment.